Why we should respect the privacy of the Taylor Remainder

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The Taylor expansion of a function (univariate or otherwise) is a widely used approach for the study of local properties of functions, estimators etc, and this note assumes familiarity with it. In applying the method, we ignore the Taylor remainder, based on the result of Taylor’s Theorem, that this remainder (the approximation error) goes to zero “really fast” as we get closer to the center of expansion.

In this note I will show graphically why the best policy towards the Taylor remainder is indeed to ignore it, especially when using a 1st-order expansion, i.e. linearization: informally put, while it can be bounded (and indeed, the proofs of many fundamental results in asymptotic theory essentially boil down to find a way to bound the remainder), it cannot be controlled inside these bounds ¹.

Consider a univariate function $f(x)$ that satisfies what needs to satisfy for the Taylor expansion to be applicable. We want to apply a 1st-order Taylor expansion with center of expansion $x_0$:

$$f(x_i) = f(x_0) + f'(x_0)(x_i - x_0) + R_i$$

We can depict this graphically as follows:

¹ There are also issues as regards convergence of the Taylor series, even for very familiar functions like the natural logarithm. See Loistl (1976) and Hlawitschka (1994) for a useful exchange in the field of Expected Utility theory.
To use geometric language, we have

\[
\begin{align*}
f'(x_0) &= \frac{B\Delta}{(x_1 - x_0)} \\
f(x_0) &= (0B) + f'(x_0)(x_1 - x_0) + R_i
\end{align*}
\]

so \( f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + R_i \) becomes

\[
(0\Gamma) = (0B) + \frac{(B\Delta)}{(x_1 - x_0)}(x_1 - x_0) + R_i
\]

\[\Rightarrow R_i = (0\Gamma) - [(0B) + (B\Delta)] = (0\Gamma) - (0\Delta)\]

\[\Rightarrow R_i = -(\Gamma\Delta)\]

negative, since the linear approximation based on \( x_0 \) overestimates the value of the function at \( x_1 \) in this case.

Fine. Now look at the following diagram:
Here, \( f(x_i) = f(x_0) + f'(x_0)(x_i - x_0) + R_i \) translates geometrically into

\[
(0\Delta) = (0B) + \frac{(B\Delta)}{(x_i - x_0)}(x_i - x_0) + R_i \Rightarrow R_i = (0\Delta) - \left[ (0B) + (B\Delta) \right] = (0\Delta) - (0\Delta)
\]

\[
\Rightarrow R_i = 0
\]

and the approximation is exact, no remainder and no approximation error.

Even in the specific example, one should be able to detect the change in the function curvature in between \( x_0 \) and \( x_1 \) in order to suspect what may happen with the remainder. Imagine doing it with more complicated functions, and especially, with multivariate ones.

The message should be clear: bound the Taylor remainder if necessary, but otherwise, respect its privacy.\(^2\)

References

\(^2\) A fresh exploration of the Lagrange form of the Remainder can be found in Kreminski (2010).