Dynamic stability for economic models: collecting some standard results (and presenting some not-so-standard ones)

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This deals with autonomous equations. I provide some not fully rigorous mathematical justification for stability criteria of single first-order differential and difference equations. I tabulate all stability criteria for 2×2 systems of differential equations. I present in detail and tabulate the criteria for saddle-path stability for 2×2 systems of difference equations, providing also the cases that are not usually seen in educational materials for economics courses, especially the case of an oscillating saddle-path. Relevance of these mathematical results to the peculiar needs of Economics is also discussed.

A. Single Equations

A.1. Differential Equations

Assume that we have a possibly non-linear, possibly non-homogeneous first order autonomous ("constant-coefficients") differential equation

\[
\frac{dx}{dt} = x(t) = g(x(t); b)
\]

where \( b \) is a vector of fixed coefficients.

Assume that the differential equation has a fixed point \( x^* \) i.e. that \( \exists x^* : \frac{dx}{dt} \bigg|_{x(t)=x^*} = g(x^*; b) = 0 \).

The necessary and sufficient condition for the fixed point to be asymptotically stable is

\[
g'(x^*; b) < 0
\]
where $g'(x^*;b)$ is the derivative of $g(x(t);b)$ with respect to $x$, evaluated at $x^*$.

**Why is that?**

Assume that at a point in time $t$, the process has the value $x(t) = x^* + \varepsilon$.

Taking a first-order Taylor expansion of the time derivative with respect to $x(t)$ with center of expansion $x^*$ we have

$$\frac{dx}{dt} = x(t) = g(x^*;b) + g'(x^*;b)(x(t) - x^*) + R_i$$

$R_i$ is the Taylor remainder. Since our interest lies in examining $g'(x^*;b)$, it means we are interested in what happens as $x(t) \to x^* \Rightarrow \varepsilon \to 0$. And in such a case $R_i \to 0$ faster than how $x(t)$ tends to $x^*$. So we can safely ignore $R_i$. Using also $g(x^*;b) = 0$, $x(t) = x^* + \varepsilon$ we can re-write the expansion

$$\dot{x}(t) \approx g'(x^*;b) \cdot \varepsilon$$

First, assume that $x(t) > x^* \Rightarrow \varepsilon > 0$. We are above the fixed point, and for stability we want to move towards it, i.e. we want the value of $x(t)$ to decrease as time passes. So we want $\dot{x}(t) < 0$. Since $\varepsilon > 0$ we must have $g'(x^*;b) < 0$.

Now assume that $x(t) < x^* \Rightarrow \varepsilon < 0$. We are below the fixed point, and for stability we want the value of $x(t)$ to increase as time passes. So we want $\dot{x}(t) > 0$. Since here we have $\varepsilon < 0$, it follows that we must have $g'(x^*;b) < 0$.

Therefore in both cases, irrespective of whether we are above or below the fixed point, we indeed require $g'(x^*;b) < 0$, for asymptotic stability.

If $g'(x^*;b) > 0$ the fixed point is unstable. The borderline case $g'(x^*;b) = 0$ (possibly interesting when we have a non-linear equation), needs deeper investigation to determine stability. Those interested can consult Part II and Mathematical Appendix B in Samuelson (1983). Some of his terminology is not used anymore, but he provides details rarely found in contemporary books.
A.2. Difference Equations

Assume that we have a possibly non-linear, possibly non-homogeneous first order autonomous difference equation

\[ x_{t+1} = f(x_t; b) \]

where \( b \) is a vector of fixed coefficients. Note that while in the previous case of the differential equation, the function \( g(x(t); b) \) measured the rate of change of the process with respect to time, here the function \( f(x_t; b) \) determines the level of the process in the next time-period.

Assume that the equation has a fixed point \( x^* \) i.e. that

\[ \exists x^* : f(x^*; b) = x^* \]

The necessary and sufficient condition for the fixed point to be asymptotically stable is

\[ \left| f'(x^*; b) \right| < 1 \]

where \( f'(x^*; b) \) is the derivative of \( f(x_t; b) \) with respect to \( x \), evaluated at \( x^* \).

How come?

Assume that in some period \( t \), the process has the value

\[ x_t = x^* + \varepsilon \Rightarrow \varepsilon = x_t - x^*. \]

The difference equation holds here too, so

\[ x_{t+1} = f(x_t; b) = f(x^* + \varepsilon; b) \]

Asymptotic stability means that if we are away from the fixed point, we move towards it. So in order to have asymptotic stability here it must be the case that \( x_{t+1} \) is "closer in distance" to \( x^* \) than \( x_t \) is. Namely we must have

\[ \frac{|x_{t+1} - x^*|}{|x_t - x^*|} < 1 \Rightarrow \frac{|x_{t+1} - x^*|}{|x_t - x^*|} < \frac{-1}{|x_t - x^*|} < 1 \]
Now, by its definition, the derivative of $f(x;b)$ at $x^*$ is

$$f'(x^*;b) = \lim_{\varepsilon \to 0} \frac{f(x^* + \varepsilon; b) - f(x^*; b)}{\varepsilon}$$

and we assume that it exists.

Using $\varepsilon = x_i - x^*$, $x_{i+1} = f(x^* + \varepsilon; b)$, $f(x^*; b) = x^*$ we can re-write this as

$$f'(x^*; b) = \lim_{x_i \to x^*} \frac{x_{i+1} - x^*}{x_i - x^*}$$

If asymptotic stability does hold then by the continuity of the limiting operation we will also have the above limit inside the $(-1,1)$ interval,

$$-1 < \frac{x_{i+1} - x^*}{x_i - x^*} < 1 \implies -1 < \lim_{x_i \to x^*} \frac{x_{i+1} - x^*}{x_i - x^*} < 1 \implies \left| \lim_{x_i \to x^*} \frac{x_{i+1} - x^*}{x_i - x^*} \right| < 1$$

$$\implies \left| f'(x^*; b) \right| < 1$$

indeed.

Moreover (a property specific to difference equations), if

$$0 < f'(x^*; b) < 1$$

the system will converge monotonically, while if

$$-1 < f'(x^*; b) < 0$$

the system will approach the fixed point by damped oscillations.

Solve a simple homogeneous linear difference equation of first order to understand why this is so.

In general, difference equations are richer in stability results ("limiting behavior of solutions") than differential equations. A convenient table summarizing all possible cases for the linear case can be found in Goldberg (1958), Table 2.2. and Figure 2.3 pp. 84-85.

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B. 2 × 2 Systems of Equations

B1. Differential Equations

Consider the 2×2 autonomous 1st order linear and non-linear systems of differential equations:

<table>
<thead>
<tr>
<th>Linear</th>
<th>Non-Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ y_1(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \gamma_1 ]</td>
<td>[ y_1(t) = f(y_1(t), y_2(t); \mathbf{q}_1) ]</td>
</tr>
<tr>
<td>[ y_2(t) = \beta_1 y_1(t) + \beta_2 y_2(t) + \gamma_2 ]</td>
<td>[ y_2(t) = g(y_1(t), y_2(t); \mathbf{q}_2) ]</td>
</tr>
</tbody>
</table>

B.1.1. Stability of the linear system

The system can be written in matrix notation

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = A \begin{bmatrix} y_1(t) \\
y_2(t)\end{bmatrix} + \Gamma, \quad A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \quad \Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}
\]

The \( \Gamma \) vector plays no role in the system’s stability properties which are assessed through the characteristic equation of the coefficient matrix \( A \):

\[
|A - \lambda I_2| = 0 \Rightarrow \begin{vmatrix} \alpha_1 - \lambda & \alpha_2 \\ \beta_1 & \beta_2 - \lambda \end{vmatrix} = 0 \Rightarrow (\alpha_1 - \lambda)(\beta_2 - \lambda) - \alpha_2 \beta_1 = 0
\]

\[
\Rightarrow \lambda^2 - (\alpha_1 + \beta_2) \lambda + (\alpha_1 \beta_2 - \alpha_2 \beta_1) = 0
\]

But this means that stability can be directly expressed in terms of the trace and the determinant of \( A \), because the characteristic equation essentially is

\[
\lambda^2 - \text{tr}(A) \lambda + \det(A) = 0, \text{ since } \text{tr}(A) = \alpha_1 + \beta_2, \text{ det}(A) = \alpha_1 \beta_2 - \alpha_2 \beta_1,
\]

and the eigenvalues/characteristic roots are calculated by
\[ \lambda_1, \lambda_2 = \frac{\text{tr}(A) \pm \sqrt{\Delta_2}}{2}, \quad \Delta_2 = \sqrt{\left(\text{tr}(A)\right)^2 - 4\det(A)} \]

Stability criteria can then be tabulated as follows:

<table>
<thead>
<tr>
<th>Conditions for stability of 2x2 system of 1st-order differential equations</th>
<th>real roots</th>
<th>complex roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{det}(A) &lt; 0 )</td>
<td>( \Delta_2 &gt; 0 )</td>
<td>( \Delta_2 = 0 )</td>
</tr>
</tbody>
</table>
| \( \Rightarrow \Delta_2 > 0 \) | Stable | Stable | Stable  
(approaching the fixed point by damped oscillations) |
| \( \text{tr}(A) < 0 \) | "Saddle-path" Stable  
(mathematically this is characterized as "unstable") | Stable | ? |
| \( \text{tr}(A) = 0 \) | Stable | ? |
| \( \text{tr}(A) > 0 \) | Unstable |

So the necessary and sufficient condition for saddle-path stability is that \( \text{det}(A) < 0 \), while if \( \text{det}(A) > 0 \), then stability depends on the sign of the trace of the matrix - and when stability obtains, it won’t be saddle-path but “asymptotically stable”.

A note of warning: as we will see in the next section, the conditions for saddle-path stability of the corresponding system of difference equations bear no resemblance to the above.

Exercise 1
Verify the correctness of the above table, by mapping it to the stability criteria expressed in terms of characteristic roots. You can find the full list of these criteria, alongside an extended discussion and detailed graphical representation that helps build intuition, in Barro & Sala-i-Martin (2004) book, 2nd ed., Mathematical Appendix A1, pp. 568-596.

Exercise 2
Ponder the question mark in the table. Is this an interesting case from the point of view of economics, or not?
Discussion
As mentioned in the table, from a mathematician's point of view "saddle-path stability" is a misnomer, since the system is considered unstable in this case. This is intuitive and correct for inanimate phenomena, since "saddle-path stability" means that there exists a unique path that leads to the fixed point - and infinite others leading away from it: how naive would we be to think that a physical system will move along this single path? The system will diverge even under the slightest perturbation (and since, even in positive sciences, models are abstractions from reality, we are certain that such perturbations will happen). Moreover, the actual path taken will start at a point depending on the initial conditions. How lucky could we be so that these initial conditions put the system on the unique saddle-path?

But when the system is used to model situations where conscious decisions are involved, then saddle-path stability is exactly what suits us: a unique path towards the fixed point, on which we will stay by constantly adjusting our behavior by a conscious decision - and if something exogenous pushes us temporarily out of this path and/or changes the path itself, we will do what we have to do in order to get back on the saddle.

On the other hand, if we obtain mathematical stability proper, it implies that no-matter how we behave economically, what decisions we take, we will arrive at the fixed point. There is a fundamental point here: economic systems are fundamentally unstable - they are maintained in equilibrium (or near it, moving towards it etc) by the conscious effort and interplay of the decisions made by humans.

Rethink now the Solow model of Growth, where the system is characterized by mathematical stability proper. In light of the previous discussion, is Solow's model flawed and useless by an economics point of view?

No. On the contrary, it reveals to us how we can obtain mathematical stability even in an economic system: by commitment, which is a special, inflexible kind of decision. The "exogenously fixed" savings rate in the Solow model (which is essentially what brings mathematical stability), is equivalent to a commitment to save and invest a specific portion of output, no matter what: we have stripped ourselves from the "freedom to choose" the savings rate.

Reversely: the more choices we have, the fewer long-term solutions remain. This may sound impressively counter-intuitive, but it's true: Rational choice is its own dictator.
B.1.2. Stability of the non-linear system

In the case of the non-linear system we need to calculate the Jacobian of the system and evaluate it at the fixed point under study:

\[
J^* = \begin{bmatrix}
\frac{\partial y_1}{\partial y_1}(y_1^*, y_2^*) & \frac{\partial y_1}{\partial y_2}(y_1^*, y_2^*) \\
\frac{\partial y_2}{\partial y_1}(y_1^*, y_2^*) & \frac{\partial y_2}{\partial y_2}(y_1^*, y_2^*)
\end{bmatrix} = \begin{bmatrix}
f_1(y_1^*, y_2^*; q_1) & f_2(y_1^*, y_2^*; q_1) \\
g_1(y_1^*, y_2^*; q_2) & g_2(y_1^*, y_2^*; q_2)
\end{bmatrix}
\]

The Jacobian evaluated at the fixed point, being a matrix of derivatives at that point, codifies how the system, currently at its fixed point, tends to move if the variables are perturbed away from their fixed-point values. In the linear case, the coefficient matrix \( A \) is essentially a Jacobian itself. When we “linearize” a system around its steady-state by a 1st-order Taylor series approximation, we essentially use as coefficients the 1st partial derivatives of the system evaluated at the fixed point, i.e. \( J^* \).

Since we have assumed that the system is autonomous (has constant coefficients), it follows that \( J^* \) will be a matrix of constants (since any variables remaining in \( J^* \) have been replaced by their fixed-point values). Then the same criteria as in the case of linear systems apply, by considering the eigenvalues of \( J^* \) (and therefore its trace and determinant), instead of those of \( A \).

Discussion

Revisit the stability criteria table. Realize that the “overall more influential” criterion for stability is whether the trace is negative (in which case some form of stability always holds), but also that a negative determinant is “strong enough” to provide saddle-path stability even when the trace is positive. Attempt to generate some intuition here.
B2. Difference Equations

We consider the linear system, while for the non-linear case, the linearization principle holds here too.
We have the system (homogeneous for simplicity, the constant terms play no role as regards stability, but in the case of linear difference equations, if constants are set to zero we essentially have normalized the fixed point to be equal to the zero vector),

\[
\begin{align*}
x_{t+1} &= \alpha_1 x_t + \alpha_2 y_t \\
y_{t+1} &= \beta_1 x_t + \beta_2 y_t
\end{align*}
\]

\[\Rightarrow \begin{bmatrix} x_{t+1} \\
y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\
y_t \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 & \alpha_2 \\
\beta_1 & \beta_2 \end{bmatrix}\]

Here too stability of the system is assessed by examining the eigenvalues \( \lambda_1, \lambda_2 \) of the matrix \( A \). We can obtain this result if we transform the system into a single second-order difference equation as follows: Lag once the two equations to obtain

\[
\begin{align*}
x_t &= \alpha_1 x_{t-1} + \alpha_2 y_{t-1} \\
y_t &= \beta_1 x_{t-1} + \beta_2 y_{t-1}
\end{align*}
\]

\[
\Rightarrow \begin{bmatrix} y_{t-1} \\
x_t \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_2} x_t - \frac{\alpha_1}{\alpha_2} x_{t-1} \\
\beta_1 x_{t-1} + \beta_2 y_{t-1} \end{bmatrix} \Rightarrow y_t = \beta_1 x_{t-1} + \beta_2 \left( \frac{1}{\alpha_2} x_t - \frac{\alpha_1}{\alpha_2} x_{t-1} \right)
\]

Substitute this into the equation for \( x_{t+1} \):

\[
x_{t+1} = \alpha_1 x_t + \alpha_2 \left( \beta_1 x_{t-1} + \beta_2 \left( \frac{1}{\alpha_2} x_t - \frac{\alpha_1}{\alpha_2} x_{t-1} \right) \right) = \alpha_1 x_t + \alpha_2 \beta_1 x_{t-1} + \beta_2 x_t - \alpha_1 \beta_2 x_{t-1}
\]

\[
\Rightarrow x_{t+1} - \left( \alpha_1 + \beta_2 \right) x_t + \left( \alpha_1 \beta_2 - \alpha_2 \beta_1 \right) x_{t-1} = 0
\]

\[
\Rightarrow x_{t+1} - \text{tr}(A) x_t + \det(A) x_{t-1} = 0
\]

This is a second-order difference equation with coefficient structure identical to the characteristic equation of matrix \( A \).

Now do the same for \( y_{t+1} \)...
\[ \begin{align*}
y_{t+1} &= \beta_1 \left[ \frac{1}{\beta_1} y_t - \frac{\beta_2}{\beta_1} y_{t-1} \right] + \alpha_2 y_{t-1} + \beta_2 y_t = \alpha_1 y_t - \alpha_1 \beta_2 y_{t-1} + \beta_1 \alpha_2 y_{t-1} + \beta_2 y_t, \\
&\Rightarrow y_{t+1} - (\alpha_1 + \beta_2) y_t + (\alpha_1 \beta_2 - \alpha_2 \beta_1) y_{t-1} = 0 \\
&\Rightarrow y_{t+1} - \text{tr}(A) y_t + \text{det}(A) y_{t-1} = 0
\end{align*} \]

...to see that it is exactly the same as the one obtained for \( x_{t+1} \).

We see that both have exactly the same characteristic equation as the 2 × 2 system, and so the same stability properties. Stability properties for a 2nd-order difference equation can be found in many books, consult for example Sydsaeter & Hammond (1995) ch. 20.

A 2nd-order linear difference equation (and so also a 2 × 2 system of 1st-order linear difference equations) is asymptotically stable if, and only if, both eigenvalues of matrix \( A \) are less than unity in absolute value, or "inside the unit disk" (and an analogous result holds when they are complex).

For reasons of comparison with what comes next, the necessary and sufficient condition for asymptotic stability expressed in terms of the trace and determinant of \( A \) is

\[
\left| \text{tr}(A) \right| < 1 + \text{det}(A) < 2 \quad \text{or analytically}
\]

\[
1 - \text{tr}(A) + \text{det}(A) > 0, \quad 1 + \text{tr}(A) + \text{det}(A) > 0, \quad 1 - \text{det}(A) > 0
\]

Note that the above imply that necessary for asymptotic stability is that \( |\text{det}(A)| < 1 \).

The books mainly focus on the criteria for asymptotic stability, with and without oscillations (when the characteristic roots are complex, if we have asymptotic stability, it will exhibit oscillations). Usually the borderline cases are left untreated (for difference equations, the borderline values are -1 and 1, and so again, they provide more limiting cases than differential equations). None of the sources I found had a detailed treatment of the case(s) of saddle-path stability (which interests Economics), so I will now provide one here.

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1 Elaydi (2005) ch. 2,3,4, provides also (some of) the borderline cases, but his book, although of undergraduate level, is not recommended for inexperienced eyes: it is directed to mathematics students and it is a bit chaotic in its structure.
B3. Saddle Path Stability for Difference Equations

For a 2×2 system of first-order difference equations, we have "saddle-path stability" only if the eigenvalues/characteristic roots are distinct, and if moreover one is inside the unit disk and the other outside.

This implies that we can obtain saddle-path behavior only if the characteristic roots are real, because if they are complex, the "one outside-one inside" condition would have to be imposed on their moduli. But in the 2×2 system, complex roots have the same modulus, so saddle-path behavior is not feasible with complex roots.

The tabulated results are as follows:

<table>
<thead>
<tr>
<th>Conditions for saddle-path stability of 2×2 system of 1st-order difference equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>real and distinct characteristic roots only</td>
</tr>
<tr>
<td>( \det (A) &lt; 0 )</td>
</tr>
<tr>
<td>( \lambda_1 &lt; -1 &lt; 0 &lt; \lambda_2 &lt; 1 )</td>
</tr>
<tr>
<td>Stable root positive, unstable root negative: monotonic adjustment</td>
</tr>
<tr>
<td>(</td>
</tr>
<tr>
<td>( + ) if ( -1 &lt; \text{tr}(A) &lt; 0 )</td>
</tr>
<tr>
<td>( \Rightarrow</td>
</tr>
<tr>
<td>( \lambda_1 &lt; -1 &lt; \lambda_2 &lt; 0 )</td>
</tr>
<tr>
<td>Both roots negative: oscillating saddle path</td>
</tr>
<tr>
<td>(</td>
</tr>
</tbody>
</table>

| \( \det (A) > 0 \)                                           |
|\( -1 < \lambda_2 < 0 < 1 < \lambda_1 \)                      |
| Stable root negative, unstable root positive: oscillating saddle path |
| \( |\text{tr}(A)| > |\det(A)| - 1, \)                       |
| \( + \) if \( 0 < \text{tr}(A) < 1 \)                      |
| \( \Rightarrow |\text{tr}(A)| > 1 - |\det(A)| \)                    |
| \( 0 < \lambda_2 < 1 < \lambda_1 \)                                 |
| Both roots positive: monotonic adjustment                      |
| \( |\text{tr}(A)| > 1 + |\det(A)| \)                        |

We can detect some relations:

a) Same-sign roots require a positive determinant.
b) The sign of the unstable root is determined by the sign of the trace.
c) The sign of the stable root is determined by the sign of the product of the trace times the determinant.
Moreover, looking at the table we can conclude that the cases where $\text{tr}(A) > 0$ can be subsumed to the cases where $\text{tr}(A) < 0$, as regards the criteria to obtain saddle-path stability. But keeping them separated permits us to know when we will have a saddle path with damped oscillations. As we will see in the numerical example, such a saddle path reflects wildly different behavior than a saddle-path without oscillations. So it is well worth the trouble to be able to distinguish between them.

A general sufficient (but not necessary) condition emerges:

If $|\text{tr}(A)| \geq 2$, $|\text{det}(A)| < 1$ the system is saddle-path stable

as well as an unstable region:

If $|\text{tr}(A)| < 1$, $|\text{det}(A)| > 2$ the system is unstable

(remember that $|\text{det}(A)| < 1$ is necessary for asymptotic stability).

These last criteria, being numerical, may not be easy to verify in an abstract model with symbolic coefficients. Nevertheless, they may be useful in estimated models, or models calibrated for simulation.

Equation-level explosiveness with system-level saddle path stability.

The sufficient criterion reveals that in a system of equations, each equation separately can have a "unit-root" or even exhibit explosive tendencies with respect to its own past. But, being in a system and affected by another variable, we may nevertheless obtain a saddle path to the fixed point.

For example, by the sufficient criterion above, the following systems are saddle-path stable:

\[
\begin{align*}
&x_{t+1} = x_t + 0.5y_t + y'_1 \\
y_{t+1} = 0.5x_t + y_t + y'_2
\end{align*}
\]

\[
\begin{align*}
&x_{t+1} = 2x_t + y_t + y'_1 \\
y_{t+1} = 3.5x_t + 2y_t + y'_2
\end{align*}
\]

In the first case, each equation separately has a unit-root as regards its own past, while in the second case it exhibits explosive tendencies. Still, there exists a unique path to the fixed-point.
Discussion
What happens if the system of equations describes an economic model? We know that we need as many "predetermined" variables as "stable roots" and as many "not-predetermined" variables as "unstable roots" (see Blanchard and Kahn 1980. Although they study difference equations, the essence of their result carries over to differential equations also).

Another way to label predetermined and not-predetermined variables is by using the terms “state” and “control/decision/jump” variables respectively. We can think of them in the following terms:

A state variable is a variable whose “law of motion” springs from its nature (or by constraints exogenous to the model), and cannot be by-passed. Its level is affected by decision variables, but only through the state variable’s law of motion.

A decision variable is a variable whose law of motion emerges out of optimal-behavior considerations, and therefore this law of motion can be ignored: its level can be directly decided, and therefore, the variable can “jump” in value, if this is the optimal think to do, instead of "following blindly" the general optimal rule derived. Such an optimal need to by-pass the general optimal rule occurs when some parameter or exogenous magnitude of the model changes. Remember also that in an economic model described by such a system, we never fix an "initial value" for the decision variable (for example consumption), but only for the state variable (say, capital). This is because the initial level of the decision variable must be free to be determined endogenously, so that the economy is placed on the saddle-path.

From another point of view, for each "unstable root", which would tend to send us to a divergent path, we need a corresponding decision variable, which "keeps in check" this tendency through purposeful behavior.

B.3.1 Saddle-path stability with an oscillating saddle-path

I will now provide a numerical example of a 2×2 system which exhibits saddle-path stability with a saddle-path characterized by damped oscillations, and in the process I will also detail how we go about in assessing the properties of such a system in general.

The steps involved are the following:
1) Determine the fixed point
2) Determine the "zero-change" loci for the two variables
3) Draw the phase diagram to qualitatively assess the dynamic properties of the system
4) Determine the equation describing the saddle-path towards the fixed point.
We consider the system

\[ \begin{align*}
    x_{t+1} &= \alpha_1 x_t + \alpha_2 y_t + \gamma_1 \\
    y_{t+1} &= \beta_1 x_t + \beta_2 y_t + \gamma_2
\end{align*} \]

\[ \Rightarrow \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \Gamma \]

1) Determine the fixed point

The fixed point must also satisfy the system above, and remain fixed. Writing it as a column vector \( z^* = (x^*, y^*)' \), it must satisfy

\[ z^* = Az^* + \Gamma \Rightarrow (I_z - A)z^* = \Gamma \Rightarrow z^* = (I_z - A)^{-1} \Gamma \]

and we assume that inversion of the matrix is feasible. In detail, this is

\[ z^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 - \alpha_1 & -\alpha_2 \\ -\beta_1 & 1 - \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \]

Performing the simple inversion, we have

\[ z^* = \begin{cases} 
    \frac{1}{(1 - \alpha_1)(1 - \beta_2) - \alpha_2 \beta_1} \begin{bmatrix} 1 - \beta_2 & \alpha_2 \\ \beta_1 & 1 - \alpha_1 \end{bmatrix} \gamma_1 \\
    \frac{1}{1 - \beta_2 - \alpha_1 + \alpha_2 \beta_2 - \alpha_2 \beta_1} \begin{bmatrix} (1 - \beta_2) \gamma_1 + \alpha_2 \gamma_2 \\ \beta_1 \gamma_1 + (1 - \alpha_1) \gamma_2 \end{bmatrix}
\end{cases} \]

\[ \Rightarrow \begin{cases} 
    x^* = \frac{(1 - \beta_2) \gamma_1 + \alpha_2 \gamma_2}{1 - \text{tr}(A) + \text{det}(A)} \\
    y^* = \frac{(1 - \alpha_1) \gamma_2 + \beta_1 \gamma_1}{1 - \text{tr}(A) + \text{det}(A)}
\end{cases} \]

2) Determine the "zero-change" loci for the two variables

We want to express a relation between the contemporaneous values of \( x_t, y_t \) such that they remain fixed. For each difference equation of the system we will obtain a different relationship. It will facilitate the next step if nevertheless we express both "zero-change" loci in terms of the one variable being a function of the other. More over I will use a weak inequality instead of an equality in order to easily determine the dynamics outside each locus. We have

\[ x_{t+1} - x_t = \Delta x_{t+1} \geq 0 \Rightarrow (\alpha_1 - 1)x_t + \alpha_2 y_t + \gamma_1 \geq 0 \Rightarrow y_t \geq \frac{1 - \alpha_1}{\alpha_2} x_t - \frac{\gamma_1}{\alpha_2} \]
Note that the inequality would have changed direction in the case where \( \alpha_2 < 0 \) (remember that!).

As an equality, the above gives a linear relation in the \((x, y)\) space of \( y \) as a function of \( x \), on which \( x_{t+1} = x_t \Rightarrow \Delta x_{t+1} = 0, \forall t \). Moreover, it tells us that if at any time \( y_t \) is higher than what the equation prescribes, \( x_t \) will tend to increase in value.

Also

\[
y_{t+1} - y_t = \Delta y_{t+1} \geq 0 \Rightarrow \beta x_t + (\beta_2 - 1) y_t + \gamma_2 \geq 0 \Rightarrow (1 - \beta_2) y_t \leq \beta x_t + \gamma_2
\]

\[
\Rightarrow y_t \leq \frac{\beta}{(1 - \beta_2)} x_t + \frac{\gamma_2}{(1 - \beta_2)}
\]

Again, note that the inequality would have changed direction in the case where \( \beta_2 > 1 \).

As an equality, the above gives a second linear relation in the \((x, y)\) space of \( y \) as a function of \( x \), on which \( y_{t+1} = y_t \Rightarrow \Delta y_{t+1} = 0, \forall t \). Moreover, it tells us that if at any time \( y_t \) is lower than what the equation prescribes, \( y_t \) will tend to diminish in value.

A Numerical Example.
Before moving to the phase diagram, it is time to introduce a specific numerical example. Consider the system

\[
\begin{bmatrix}
x_{t+1} \\
y_{t+1}
\end{bmatrix} = \begin{bmatrix} 0.5 & 0.9 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}
\]

We have \( \text{tr}(A) = 0.5 + 0.4 = 0.9 \), \( \text{det}(A) = 0.5 \times 0.4 - 0.9 \times 0.5 = 0.2 - 0.45 = -0.25 \)

Looking at the table with the saddle-path stability criteria, we are at the lower left corner, with positive trace and negative determinant. Moreover, the trace is smaller than unity. So for saddle path stability we must have

\[
\text{tr}(A) > |\text{det}(A)| - 1 \Rightarrow \{0.9 > |0.25| - 1\} \cap \{0.9 > 1 - |0.25|\}
\]

Both conditions hold, so the table tells us that we have saddle-path stability.

The characteristic roots of the system are
\[ \lambda_1, \lambda_2 = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}(A))^2 - 4\det(A)}}{2} = \frac{0.9 \pm \sqrt{(0.9)^2 - 4 \times (-0.25)}}{2} \]

\[ = \frac{0.9 \pm \sqrt{1.81}}{2} \approx \left\{ \begin{array}{l} 1.12 \\ -0.22 \end{array} \right. \]

So indeed we have one positive unstable root greater than unity, and one negative stable root. Since the stable root is negative, we expect oscillations.

The fixed point for the specific coefficients is

\[ \left\{ \begin{array}{l} x^* = \frac{(1 - 0.4) \times 0.5 + 0.9 \times (-1.0)}{1 - 0.9 - 0.25} \\ y^* = \frac{(1 - 0.5)(-1.0) + 0.5 \times 0.5}{1 - 0.9 - 0.25} \end{array} \right. \]

\[ \approx \left\{ \begin{array}{l} x^* = \frac{0.3 - 0.9}{-0.15} \\ y^* = \frac{-0.5 + 0.25}{-0.15} \end{array} \right. \]

\[ \approx \left\{ \begin{array}{l} x^* = 4 \\ y^* = 5/3 \end{array} \right. \]

Turning to the "zero-change" loci (no need to reverse the inequality signs for this particular example), we have

\[ \Delta x_{t+1} \geq 0 \Rightarrow y_t \geq \frac{1 - 0.5}{0.9} x_t - \frac{0.5}{0.9} \Rightarrow y_t = \frac{5}{9} x_t = \frac{5}{9} \]

\[ \Delta y_{t+1} \geq 0 \Rightarrow y_t \leq \frac{0.5}{(1 - 0.4)} x_t + \frac{-1}{(1 - 0.4)} \Rightarrow y_t \leq \frac{5}{6} x_t - \frac{3}{2} \]

Now we can...
3) Draw the phase diagram of the system:

![Phase Diagram]

The picture seems pretty familiar: there are two opposite regions from which we move away from the fixed point, and two others from which we move towards the fixed point along a unique saddle path. But how do we move in this last case? The usual situation depicted is for a monotonic movement: we don’t leave the region where we started.

The mathematical results tell us that the stable root here is negative, and that this means "an oscillating saddle path". Can this oscillation be "smooth" in the sense of "circling" around the fixed point, as would be the case with an asymptotically stable system with oscillations? In that situation, we would move from one region to its adjacent, and then to the adjacent, etc, getting in this circular way closer and closer to the fixed point.

But we cannot have that here. We cannot find ourselves in any of the two divergent regions, because we will diverge. So "oscillating saddle path" means jumping from one convergent region to another, and back again, although in a position closer to the fixed point: Something like
The "path" is the sequence of these discrete dots, numbered consecutively as time passes. The system moves from 1 to 2, from 2 to 3, etc. This behavior is only feasible in discrete time settings.

4) Determine the equation describing the saddle-path towards the fixed point.

Without explaining why (review matrix algebra if needed), in order to determine the saddle-path expression we have first to find the stable eigenvector $v_2$ of the system, using the stable root $\lambda_2$ and solving the equation

$$ (A - \lambda_2 I_2) v_2 = 0 \Rightarrow \begin{bmatrix} 0.5 - \lambda_2 & 0.9 \\ 0.5 & 0.4 - \lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow \begin{cases} (0.5 - \lambda_2) v_{21} + 0.9 v_{22} = 0 \\ 0.5 v_{21} + (0.4 - \lambda_2) v_{22} = 0 \end{cases} $$

Equating the left-hand sides, we have

$$(0.5 - \lambda_2) v_{21} + 0.9 v_{22} = 0.5 v_{21} + (0.4 - \lambda_2) v_{22} \Rightarrow (0.5 + \lambda_2) v_{22} = \lambda_2 v_{21}$$

$$\Rightarrow \frac{v_{22}}{v_{21}} = \frac{\lambda_2}{0.5 + \lambda_2}$$
We arrived at this ratio, because the saddle-path expression is then determined by

\[
\frac{y_t - y^*}{x_t - x^*} = \frac{v_{22}}{v_{21}} \Rightarrow y_t = y^* + \frac{v_{22}}{v_{21}}(x_t - x^*)
\]

Note that we put the second element of the eigenvector on the numerator, and the second variable of the system in the numerator also.

For our numerical example, this gives the saddle path

\[
y_t = \frac{5}{3} + \frac{\lambda_2}{0.5 + \lambda_2} (x_t - 4), \quad \lambda_2 = \frac{0.9 - \sqrt{1.81}}{2}
\]

Assume now that we have \(x_0 = 1\). Then to place the system on the saddle path, \(y_0\) must be free to be determined by the saddle path expression. If we plug these two values into the 2×2 system, and let it run, we will observe that the system moves towards the fixed point \((x^*, y^*) = (4, 5/3), 5/3 = 1.6\). The numerical values are

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>4.076</td>
</tr>
<tr>
<td>1</td>
<td>4.668</td>
<td>1.130</td>
</tr>
<tr>
<td>2</td>
<td>3.851</td>
<td>1.786</td>
</tr>
<tr>
<td>3</td>
<td>4.033</td>
<td>1.640</td>
</tr>
<tr>
<td>4</td>
<td>3.993</td>
<td>1.673</td>
</tr>
<tr>
<td>5</td>
<td>4.002</td>
<td>1.665</td>
</tr>
</tbody>
</table>

and we see that both variables oscillate as time passes, each closer and closer to their own fixed point. And when the one is above its own fixed point value, the other is below its fixed point value. This produces the "jumping" behavior shown in the phase diagram previously.
B.3.2. Technical Appendix

Derivation of the conditions for saddle-path stability in $2 \times 2$ systems of 1st order difference equations.

To find criteria in terms of the trace and determinant of $A$ we again examine the characteristic equation of the matrix

$$\lambda^2 + [−\text{tr}(A)]\lambda + \det(A) = 0,$$

with $\text{tr}(A) = \alpha_1 + \beta_2$, $\det(A) = \alpha_1\beta_2 - \alpha_2\beta_1$,

the roots of which are calculated by

$$\lambda_1, \lambda_2 = \frac{\text{tr}(A) \pm \sqrt{\Delta}}{2}, \quad \Delta = (\text{tr}(A))^2 - 4\det(A)$$

For compactness I will write in intermediate calculations $\text{tr}(A) \equiv \tau$ and $\det(A) \equiv d$

Since the roots must be distinct and real a **first general necessary condition** is that

$$\tau^2 > 4d$$

Since we want one root inside the unit disk and one outside, it follows that their absolute values cannot be equal. This immediately provides a **second general necessary condition for saddle-path stability**

$$\tau \neq 0$$

because otherwise the two roots would be symmetric around zero, and so they would be equal in absolute value.

We will see that these conditions are superseded by the necessary and sufficient conditions for the four cases that we will consider. $\lambda_i$ denotes the root that has absolute value greater than 1 (the "unstable root").
S.1. Both roots positive  

Since \( \lambda_1, \lambda_2 = \frac{\tau \pm \sqrt{\tau^2 - 4d}}{2} \),

to have both roots positive, it must be the case that

\( \tau > \sqrt{\tau^2 - 4d} > 0 \Rightarrow \tau, d > 0 \)

Then with both roots positive, if the condition for saddle path stability is to hold (one root greater than unity, the other smaller), we must have that

\( (\lambda_2 - 1)(\lambda_1 - 1) < 0 \Rightarrow \lambda_2 \lambda_1 - (\lambda_2 + \lambda_1) + 1 < 0 \)

From the general relations as regards the roots of any quadratic polynomial, we get

\( \lambda_2 \lambda_1 = d, \lambda_2 + \lambda_1 = \tau \)

So \( \lambda_2 \lambda_1 - (\lambda_2 + \lambda_1) + 1 < 0 \Rightarrow d - \tau + 1 < 0 \Rightarrow \tau > 1 + d \)

This also covers the general necessary conditions, as can be easily checked. So we conclude:

For saddle-path stability with both roots positive \( 0 < \lambda_2 < 1 < \lambda_1 \), necessary and sufficient conditions are

\( \det(A) > 0, \ \text{tr}(A) > 1 + \det(A) \)

This is perhaps the most frequent case encountered in economic models.

S.2. Both roots negative  

\( \lambda_1 < -1 < \lambda_2 < 0 \)

We have

\( \frac{\tau - \sqrt{\tau^2 - 4d}}{2} < -1 < \frac{\tau + \sqrt{\tau^2 - 4d}}{2} < 0 \Rightarrow \left| \tau - \sqrt{\tau^2 - 4d} \right| > \left| \tau + \sqrt{\tau^2 - 4d} \right| \)

This is easily seen to require the condition \( \tau < 0 \).
Further, starting from the rightmost inequality we need
\[ \tau + \sqrt{\tau^2 - 4d} < 0 \Rightarrow \sqrt{\tau^2 - 4d} < |\tau| \Rightarrow \tau^2 - 4d < \tau^2 \Rightarrow d > 0 \]

From the middle inequality we have
\[ -1 < \frac{\tau + \sqrt{\tau^2 - 4d}}{2} \Rightarrow \sqrt{\tau^2 - 4d} > -(2 + \tau) \]

while from the leftmost inequality we have
\[ \frac{\tau - \sqrt{\tau^2 - 4d}}{2} < -1 \Rightarrow \sqrt{\tau^2 - 4d} > 2 + \tau \]

The last two combined, they impose the condition
\[ \sqrt{\tau^2 - 4d} > |2 + \tau| \Rightarrow \tau^2 - 4d > 4 + 4\tau + \tau^2 \Rightarrow |\tau| > 1 + d . \]

For saddle-path stability with both roots negative \( \lambda_1 < -1 < \lambda_2 < 0 \), necessary and sufficient conditions are
\[ \text{tr}(A) < 0, \quad \text{det}(A) > 0 , \quad |\text{tr}(A)| > 1 + \text{det}(A) \]

This setup has a saddle-path to the fixed point that exhibits damped oscillations (due to the negative stable root).

**S.3. Stable root positive, unstable root negative** \( \lambda_1 < -1 < 0 < \lambda_2 < 1 \)

We have
\[ \frac{\tau - \sqrt{\tau^2 - 4d}}{2} < -1 < \tau + \frac{\sqrt{\tau^2 - 4d}}{2} < 1 \Rightarrow \left| \tau - \sqrt{\tau^2 - 4d} \right| > \left| \tau + \sqrt{\tau^2 - 4d} \right| \]

This again imposes the condition \( \tau < 0 \).

From the positivity inequality we have
\[
\frac{\tau + \sqrt{\tau^2 - 4d}}{2} > 0 \Rightarrow \sqrt{\tau^2 - 4d} > |\tau| \Rightarrow \tau^2 - 4d > \tau^2 \Rightarrow d < 0
\]

From the rightmost inequality we have

\[
\frac{\tau + \sqrt{\tau^2 - 4d}}{2} < 1 \Rightarrow \sqrt{\tau^2 - 4d} < 2 - \tau \Rightarrow \tau^2 - 4d < 4 - 4\tau + \tau^2 \Rightarrow \tau < 1 + d
\]
\[
\Rightarrow -d < 1 - \tau \Rightarrow |\tau| > |d| - 1
\]

Finally from the leftmost inequality we have

\[
\frac{\tau - \sqrt{\tau^2 - 4d}}{2} < -1 \Rightarrow \sqrt{\tau^2 - 4d} > 2 + \tau.
\]

If \(\tau < -2\) this holds. If \(\tau > -2\) then squaring we must have

\[
\tau^2 - 4d > 4 + 4\tau + \tau^2 \Rightarrow -d > 1 + \tau \Rightarrow -\tau > 1 + d \Rightarrow |\tau| > 1 - |d|
\]

If \(\tau < -1\) this will hold always.

For saddle-path stability with stable root positive, unstable root negative \(\lambda_1 < -1 < 0 < \lambda_2 < 1\) necessary and sufficient conditions are

\[
\text{tr}(A) < 0, \quad \det(A) < 0, \quad |\text{tr}(A)| > |\det(A)| - 1,
\]

and if \(-1 < \text{tr}(A) < 0 \Rightarrow |\text{tr}(A)| > 1 - |\det(A)|\) in addition

S.4. Stable root negative, unstable root positive \(-1 < \lambda_2 < 0 < 1 < \lambda_1\)

\[
-1 < \frac{\tau - \sqrt{\tau^2 - 4d}}{2} < 0 < 1 < \frac{\tau + \sqrt{\tau^2 - 4d}}{2} \Rightarrow \left|\tau - \sqrt{\tau^2 - 4d}\right| < \left|\tau + \sqrt{\tau^2 - 4d}\right|
\]

This imposes the condition \(\tau > 0\).

From the negativity inequality we have
\[
\frac{\tau - \sqrt{\tau^2 - 4d}}{2} < 0 \Rightarrow \sqrt{\tau^2 - 4d} > \tau \Rightarrow \tau^2 - 4d > \tau^2 \Rightarrow d < 0
\]

From the leftmost inequality we have
\[
-1 < \frac{\tau - \sqrt{\tau^2 - 4d}}{2} \Rightarrow \sqrt{\tau^2 - 4d} < 2 + \tau \Rightarrow \tau^2 - 4d < 4 + 4\tau + \tau^2 \Rightarrow -d - 1 < \tau
\]
\[
\Rightarrow \tau > |d| - 1
\]

From the rightmost inequality we have
\[
1 < \frac{\tau + \sqrt{\tau^2 - 4d}}{2} \Rightarrow \sqrt{\tau^2 - 4d} > 2 - \tau
\]
If \( \tau > 2 \) this holds. If \( \tau < 2 \), squaring we have
\[
\tau^2 - 4d > 4 - 4\tau + \tau^2 \Rightarrow \tau > 1 + d = 1 - |d|
\]
If \( 1 < \tau < 2 \), this will hold always.

For saddle-path stability with stable root negative, unstable root positive
\(-1 < \lambda_2 < 0 < 1 < \lambda_1 \) necessary and sufficient conditions are
\[
\text{tr}(A) > 0, \quad \text{det}(A) < 0, \quad \text{tr}(A) > |\text{det}(A)| - 1,
\]
and if \( \text{tr}(A) < 1 \Rightarrow \text{tr}(A) > 1 - |\text{det}(A)| \) in addition

References

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