Stability of $2 \times 2$ Systems of Differential Equations
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This is valid for $2 \times 2$ Constant-Coefficient (“Autonomous”) 1st order Linear and non-Linear (non-linear in the variables but linear in the derivatives) Systems of differential equations, i.e. of systems like the following:

<table>
<thead>
<tr>
<th>Linear</th>
<th>Non-Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{y}_1 = \alpha_1 y_1(t) + \alpha_2 y_2(t) + \gamma_1$</td>
<td>$\dot{y}_1 = f_1(y_1(t), y_2(t))$</td>
</tr>
<tr>
<td>$\dot{y}_2 = \beta_1 y_1(t) + \beta_2 y_2(t) + \gamma_2$</td>
<td>$\dot{y}_2 = f_2(y_1(t), y_2(t))$</td>
</tr>
</tbody>
</table>

**Note:** Results do not translate immediately for systems of difference equations.

**Stability of the Linear System**

The system can be written in matrix notation

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = A \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \Gamma, \quad A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Stability can be directly assessed by calculating the trace and the determinant of the coefficient matrix $A$. This is because the characteristic equation from which we can derive its eigenvalues and determine stability essentially is
\[
\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0, \quad \text{with} \quad \text{tr}(A) = \alpha_1 + \beta_2, \quad \det(A) = \alpha_1\beta_2 - \alpha_2\beta_1,
\]

and the eigenvalues are calculated by

\[
\lambda_1, \lambda_2 = \frac{\text{tr}(A) \pm \sqrt{\left(\text{tr}(A)\right)^2 - 4\det(A)}}{2}, \quad \Delta = \sqrt{\left(\text{tr}(A)\right)^2 - 4\det(A)}
\]

Stability criteria can then be tabulated as follows:

<table>
<thead>
<tr>
<th></th>
<th>real roots</th>
<th>complex roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \det(A) &lt; 0 )</td>
<td>( \Delta &gt; 0 )</td>
<td>( \Delta = 0 )</td>
</tr>
<tr>
<td>( \Rightarrow \Delta &gt; 0 )</td>
<td>( \Delta &gt; 0 )</td>
<td>( \Delta &lt; 0 )</td>
</tr>
<tr>
<td>( \text{tr}(A) &lt; 0 )</td>
<td>Stable</td>
<td>Stable (approaching the fixed point by damped oscillations)</td>
</tr>
<tr>
<td>( \text{tr}(A) = 0 )</td>
<td>Stable</td>
<td>Merry-go-round (*)</td>
</tr>
<tr>
<td>( \text{tr}(A) &gt; 0 )</td>
<td>Unstable</td>
<td></td>
</tr>
</tbody>
</table>

(*) "Merry-go-round" = stable orbit around the fixed point.

So the necessary and sufficient condition for saddle-path stability is that \( \det(A) < 0 \), while if \( \det(A) > 0 \), then stability depends on the sign of the trace of the matrix \( A \) (and when stability obtains, it won’t be saddle-path, but stronger – “asymptotically stable”).

Reflect on the formula for the calculation of the eigenvalues, in order to understand why the standard criteria regarding stability, expressed in terms of whether the eigenvalues are positive, negative or both, translate into the above table.
**Stability of the non-Linear System**

In the case of the non-linear system we need to calculate the Jacobian of the system and evaluate it at the fixed point under study:

\[
J^* = \begin{bmatrix}
\frac{\partial \dot{y}_1}{\partial y_1^*} (y_1^*, y_2^*) & \frac{\partial \dot{y}_1}{\partial y_2^*} (y_1^*, y_2^*) \\
\frac{\partial \dot{y}_2}{\partial y_1^*} (y_1^*, y_2^*) & \frac{\partial \dot{y}_2}{\partial y_2^*} (y_1^*, y_2^*)
\end{bmatrix}
\]

The Jacobian evaluated at the fixed point, being a matrix of derivatives at that point, codifies how the system, currently at its fixed point, tends to move if the variables are perturbed away from their fixed-point values. In the linear case, the “coefficient matrix” is essentially a Jacobian itself (i.e. as usual the non-linear case is the more general one). When we “linearize” a system around its steady-state by a 1st-order Taylor series approximation, we essentially use as coefficients the 1st partial derivatives of the system evaluated at the fixed point, i.e. \(J^*\).

Since we have assumed that the system is autonomous (has constant coefficients), it follows that \(J^*\) will be a matrix of constants (since any variables remaining in \(J^*\) have been replaced by their fixed-point values). Then the same criteria as in the case of linear systems apply, by considering the eigenvalues of \(J^*\) (and therefore its trace and determinant), instead those of \(A\).

**A trick question (but insightful)**

The system

\[
\begin{align*}
\dot{y}_1 &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \\
\dot{y}_2 &= \beta_1 y_1(t) + \beta_2 y_2(t)
\end{align*}
\]

is the same as

\[
\begin{align*}
\dot{y}_1 &= \alpha_2 y_2(t) + \alpha_1 y_1(t) \\
\dot{y}_2 &= \beta_2 y_2(t) + \beta_1 y_1(t)
\end{align*}
\]

since we have only changed the order in which we write the additive terms. But if we write them in matrix notation we will obtain two different coefficient matrices, with different trace,
determinant and eigenvalues… In fact, the determinant of the second version will be the negative of the determinant of the original version. Therefore we will arrive at different (and even, opposite) conclusions regarding the stability of the system around its fixed points. But the system is the same in both cases. What is going on here?

Once you have answered (intuitively) this question, revisit the stability criteria table. Realize that the “overall more influential” criterion for stability is whether the trace is negative (in which case some form of stability always holds), but also that a negative determinant is “strong enough” to provide saddle-path stability even when the trace is positive. Discuss the intuition here. Also, it appears that a negative trace, and a negative determinant are the two cases that produce stability. Then, one could think that the strongest version of stability would hold when both are present. But when both are present, we have only saddle-path stability. Discuss.

**Economics**

The above give the mathematical approach on the issue. But what happens if the system of equations describes an economic model? We know that we need a relation between “stable roots” and “unstable roots” (i.e. eigenvalues negative or positive) on the one hand, and “predetermined” and “not-predetermined” variables on the other (see Blanchard and Kahn 1980. Although they study difference equations, their results carry over to differential equations).

Remember that another way to label predetermined and not-predetermined variables is by using the terms “state” and “control/decision/“jump” variables. We can think of them in the following terms:

A state variable is a variable that its differential equation (its “law of motion”) springs from its nature (or by constraints exogenous to the model) and cannot be by-passed. Its current level may be affected by control/decision/“jump” variables, but only through the state variable’s law of motion.
A decision variable is a variable that its law of motion emerges out of optimal-behaviour considerations, and therefore it can be ignored (its level can be directly decided, and therefore, the variable can “jump” in value).

Therefore: a $2 \times 2$ system of differential equations can be studied as a mathematical object, and we may arrive at the conclusion that it possesses the saddle-path stability property. This means that it is structurally able to provide a unique path to the fixed-point (the “steady-state”) if, additionally, we have one decision variable (and so one state variable). Thankfully, in a $2 \times 2$ economic model, absence of decision variables makes the case uninteresting, while absence of state variables makes the case essentially non-dynamic (although it may evolve through time). So in the $2 \times 2$ case in economics, expect always to have one decision and one state variable.

References